

Gauge Fixing in Higher Derivative Gravity

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Abstract

Linearized four-derivative gravity with a general gauge fixing term is considered. By a Legendre transform and a suitable diagonalization procedure it is cast into a second-order equivalent form where the nature of the physical degrees of freedom, the gauge ghosts, the Weyl ghosts, and the intriguing “third ghosts”, characteristic to higher-derivative theories, is made explicit. The symmetries of the theory and the structure of the compensating Faddeev-Popov ghost sector exhibit non-trivial peculiarities.

Introduction

Theories of gravity with terms of any order in curvatures arise as part of the low energy effective theories of the strings [1] and from the dynamics of quantum fields in a curved spacetime background [2]. Theories of second order (four-derivative theories in the following) have been studied more closely in the literature because they are renormalizable [3] in four dimensions. This property spurred Renormalization–Group studies [4–8], including attempts to get rid of the Weyl ghosts (also known as “poltergeists”) usually occurring in higher-derivative (HD) theories. On the practical side, HD gravity greatly affects the effective potential and phase transitions of scalar fields in curved space-time, with a wealth of astrophysical and cosmological properties [9]. These phenomenological applications contributed to keep alive the theoretical interest, as illustrated by the most comprehensive introductory study available [10].

Besides the renormalization properties [3], all that was known about the structure of the (classical) theory was the particle contents, as read out of the linear decomposition of the HD propagator into pieces with second order (particle) poles. Some related aspects of the equations of motion [11] were also elucidated. Definite theoretical progress came from a procedure, based on the Legendre transformation, devised to recast four-derivative gravity as an equivalent theory of second differential order [12]. A suitable diagonalization of the resulting theory was found later [13] that yields the explicit independent fields for the degrees of freedom (DOF) involved (usually including massive Weyl ghosts), thus completing the order-reducing procedure. One should notice that theories with terms of higher order in curvatures have the same DOF and propagators of the four-derivative one, since the higher terms do not contribute to the linearized theory.

An alternative order-reducing method has been proposed [14] that introduces an auxiliary field coupled to the Einstein tensor $G_{\mu\nu}$ (or to the scalar curvature R) and featuring a squared term. It can be shown that this method is equivalent to the one based on a Legendre transformation with respect to $G_{\mu\nu}$ (or to R), the auxiliary field being a redefinition of the “momentum” conjugate to them.

The studies [12–14] above were carried out for the (non quantizable) Diff-invariant theory. An exploration of the method in the presence of gauge fixing terms has been done in a simplified HD gauge field model [15]. In this paper we implement this procedure for four-derivative gravity.

Amongst a crowd of positive and negative norm, gauge-independent and gauge ghost, massless and massive states, the famous “third ghosts” arise. These subtle ghosts, missed in [4] and properly accounted for in [5] and there since, first emerged from a functional determinant in the context of Path Integral quantization. Now they appear as the poltergeist partners of the usual gauge ghosts.

In Section 1 we present our starting Diff-invariant four-derivative theory. A very general gauge fixing term is then introduced that includes the most used ones as particular cases. Being interested in the propagators and in the ensuing DOF identification, we focus mainly on the free part of the Lagrangian. Self-interactions and interactions with other matter fields are embodied in a source term and may be treated perturbatively. Then the relevant total gauge fixed linearized theory is worked

out. Section 2 deals with the order-reducing procedure that leads to the diagonalized second-derivative equivalent theory. The structure of the propagators and the identification of the DOF is then worked out in Section 3. The Faddeev–Popov (FP) compensating Lagrangian is studied in Section 4, where an order-reduction of the fermion sector is also carried out. Particular attention is paid here to the identification of poles and to the striking cancellation mechanism of ghost loop contributions. Related to this, a discussion of the BRST symmetries involved is also made. The above results are summarized and further elucidated in the conclusion.

The definitions of the spin projectors and related formulae, the basis of local differential operators, and the notations and conventions used throughout the paper have been collected in Appendix I in order to render the work almost self-contained and more readable. Secondary calculations regarding the conditions from locality on the gauge-fixing parameters and the order-reduction of the HD fermionic FP Lagrangian have been respectively moved to two Appendices.

1 The Linearized Lagrangian

We consider a general four-derivative theory of gravity

$$\mathcal{L}_{HD} = \mathcal{L}_{inv} + \mathcal{L}_g + \mathcal{L}_m \quad , \quad (1)$$

where \mathcal{L}_m is the coupling with matter,

$$\mathcal{L}_{inv} = \sqrt{g} [aR + bR^2 + cR_{\mu\nu}R^{\mu\nu}] \quad , \quad (2)$$

is the most general Diff-invariant gravitational Lagrangian of second order in curvatures (the squared Riemann tensor has not been considered as long as a topologically trivial 4D spacetime is assumed so that the Gauss-Bonnet identity holds), and

$$\mathcal{L}_g = \sqrt{g} \frac{1}{2} \chi^\mu[h] \mathcal{G}_{\mu\nu} \chi^\nu[h] \quad , \quad (3)$$

where

$$\chi^\mu[h] \equiv A^\mu - \lambda D^\mu h \quad , \quad (4)$$

$$\mathcal{G}_{\mu\nu} \equiv \xi_1 D^\rho D_\rho g_{\mu\nu} - \xi_2 \frac{1}{2} D_{(\mu} D_{\nu)} + \xi_3 g_{\mu\nu} + \xi_4 R_{\mu\nu} + \xi_5 R g_{\mu\nu} \quad , \quad (5)$$

is a general gauge fixing term that depends on six gauge parameters and generally contains HD as well as lower-derivative (LD) terms. One may obtain the gauge fixings used in [4]–[8] by specializing these parameters.

In order to study the propagating DOF of the theory we work the quadratic terms in $h_{\mu\nu}$ out of \mathcal{L}_{HD} . Dropping total derivatives, they write

$$\mathcal{L}_{HD} = \mathcal{L}_{inv}^{(2)} + \mathcal{L}_g^{(2)} + \mathcal{L}_s = \frac{1}{2} h^{\mu\nu} (P_{\mu\nu,\rho\sigma}^{inv} + P_{\mu\nu,\rho\sigma}^g) h^{\rho\sigma} + \mathcal{L}_s \quad . \quad (6)$$

The source term \mathcal{L}_s includes the interactions with matter fields ϕ and all the self-interactions of $h_{\mu\nu}$ affected by the Newton constant G_N . Here and in the following

the indices are raised and lowered by $\eta_{\mu\nu}$ and usually omitted for simplicity whenever no ambiguity arises.

The differential operator kernel for the diff-invariant part is

$$P^{inv} = a\Box \left[\frac{1}{2}P^{(2)} - P^{(S)} \right] + 6b\Box^2 P^{(S)} + c\Box^2 \left[\frac{1}{2}P^{(2)} + 2P^{(S)} \right] . \quad (7)$$

The gauge fixing contribution

$$\mathcal{L}_g^{(2)} = \frac{1}{2}(A^\mu - \lambda\partial^\mu h)(\xi_1\Box\eta_{\mu\nu} - \xi_2\partial_\mu\partial_\nu + \xi_3\eta_{\mu\nu})(A^\nu - \lambda\partial^\nu h) \quad (8)$$

yields

$$\begin{aligned} P^g = & -\Box\lambda^2[(\xi_1 - \xi_2)\Box + \xi_3](3P^{(S)} + P^{(W)} + P^{\{SW\}}) \\ & + \xi_2\Box^2 P^{(W)} - \Box[\xi_1\Box + \xi_3]\left(\frac{1}{2}P^{(1)} + P^{(W)}\right) \\ & + \lambda\Box[(\xi_1 - \xi_2)\Box + \xi_3](2P^{(W)} + P^{\{SW\}}) . \end{aligned} \quad (9)$$

One recognizes in (8) the linearized $\chi^\mu[h]$ and the h -independent part of $\mathcal{G}_{\mu\nu}$, which we call $\mathcal{G}^{(h)}$ in the following.

Thus the complete HD differential operator kernel is

$$\begin{aligned} P &= P^{inv} + P^g \\ &= \frac{1}{2}\Box(c\Box + a)P^{(2)} - \frac{1}{2}\Box(\xi_1\Box + \xi_3)P^{(1)} \\ &\quad + \Box[-a + 2(3b + c)\Box - 3\lambda^2((\xi_1 - \xi_2)\Box + \xi_3)]P^{(S)} \\ &\quad - (\lambda - 1)^2\Box((\xi_1 - \xi_2)\Box + \xi_3)P^{(W)} \\ &\quad - \lambda(\lambda - 1)\Box((\xi_1 - \xi_2)\Box + \xi_3)P^{\{SW\}} . \end{aligned} \quad (10)$$

By decomposing P in its HD and LD parts, namely

$$P = M\Box^2 + N\Box , \quad (11)$$

where

$$\begin{aligned} M &\equiv \frac{c}{2}P^{(2)} - \frac{1}{2}\xi_1P^{(1)} \\ &\quad + \left(2(3b + c) - 3\lambda^2(\xi_1 - \xi_2)\right)P^{(S)} - (\lambda - 1)^2(\xi_1 - \xi_2)P^{(W)} \\ &\quad - \lambda(\lambda - 1)(\xi_1 - \xi_2)P^{\{SW\}} \\ N &\equiv \frac{1}{2}aP^{(2)} - \frac{1}{2}\xi_3P^{(1)} \\ &\quad - \left(a + 3\lambda^2\xi_3\right)P^{(S)} - (\lambda - 1)^2\xi_3P^{(W)} - \lambda(\lambda - 1)\xi_3P^{\{SW\}} , \end{aligned} \quad (12)$$

equation (6) may be written as

$$\mathcal{L}_{HD} = \frac{1}{2}h\Box(M\Box + N)h + \mathcal{L}_s . \quad (13)$$

Dropping total derivatives, it can be given the more convenient form

$$\mathcal{L}_{HD}[h, \square h] = \frac{1}{2}(\square h)M(\square h) + \frac{1}{2}hN(\square h) + \mathcal{L}_s \quad . \quad (14)$$

The HD Euler's equation takes also the form

$$\square(M\square + N)^{\mu\nu,\rho\sigma}h_{\rho\sigma} = T^{\mu\nu} \quad , \quad (15)$$

where $T^{\mu\nu} \equiv -\frac{\delta\mathcal{L}_s}{\delta h_{\mu\nu}}$.

2 Second order equivalent theory

In order to perform a Lorentz-covariant Legendre transformation [13][16] of our HD Lagrangian, the form of (14) trivially suggests defining the conjugate variable

$$\pi^{\mu\nu} = \frac{\partial\mathcal{L}_{HD}}{\partial(\square h_{\mu\nu})} \quad . \quad (16)$$

One finds

$$\pi = M(\square h) + \frac{1}{2}Nh + O(G_N) \quad , \quad (17)$$

where the contributions from the gravitational interactions may be accounted for perturbatively in G_N , or may be simply ignored for the analysis of the propagating DOF.

As required, (17) is invertible and gives

$$\square h = M^{-1} \left[\pi - \frac{1}{2}Nh \right] \equiv F[h, \pi] \quad . \quad (18)$$

Notice that the operators M and N are invertible as long as gauge fixing terms have been introduced. Otherwise they would project into the spin-state subspace $2 \oplus S$, then being singular.

The Lorentz-covariant Hamiltonian-like function is then

$$\begin{aligned} \mathcal{H}[h, \pi] &= \pi F[h, \pi] - \mathcal{L}_{HD}[h, F[h, \pi]] \\ &= \frac{1}{2} \left[\frac{1}{2}Nh - \pi \right] M^{-1} \left[\frac{1}{2}Nh - \pi \right] - \mathcal{L}_s \quad . \end{aligned} \quad (19)$$

The equations of motion turn out to be the system of canonical-like equations

$$\square h = \frac{\partial\mathcal{H}}{\partial\pi} \quad (20)$$

$$\square\pi = \frac{\partial\mathcal{H}}{\partial h} \quad . \quad (21)$$

The familiar negative sign one would expect in (21) is absent because the definition (16) involves second derivatives of the field h instead of the usual velocity [15].

They may also be derived by a variational principle from the so called (now second-derivative) Helmholtz Lagrangian

$$\mathcal{L}_H[h, \pi] = \pi \square h - \mathcal{H}[h, \pi] \quad . \quad (22)$$

In fact from (22) one sees that (20) is the Euler's equation for π and (21) is the one for h . From (20) (which is nothing but equation (18)) one may work out π as given by (17). Substituting it in (21) one recovers (15), namely the original HD equation of motion.

Mixed $\pi - h$ terms occur in (22). The diagonalization can be performed by defining new tilde fields such that

$$\begin{aligned} h &= \tilde{h} + \tilde{\pi} \\ \pi &= \frac{N}{2}(\tilde{h} - \tilde{\pi}) \quad . \end{aligned} \quad (23)$$

or conversely

$$\begin{aligned} \tilde{h} &= N^{-1} \left[\frac{1}{2} N h + \pi \right] \\ \tilde{\pi} &= N^{-1} \left[\frac{1}{2} N h - \pi \right] \quad . \end{aligned} \quad (24)$$

Then \mathcal{L}_H finally becomes the desired LD theory

$$\mathcal{L}_{LD} = \frac{1}{2} \tilde{h} N \square \tilde{h} - \frac{1}{2} \tilde{\pi} (N \square + N M^{-1} N) \tilde{\pi} + \mathcal{L}_s \quad , \quad (25)$$

where

$$\begin{aligned} N M^{-1} N &= \frac{a^2}{2c} P^{(2)} - \frac{\xi_3^2}{2\xi_1} P^{(1)} \\ &+ \frac{a^2(\xi_1 - \xi_2) - 3\lambda^2 \xi_3^2 2(3b + c)}{2(3b + c)(\xi_1 - \xi_2)} P^{(s)} \\ &- \frac{(\lambda - 1)^2 \xi_3^2}{\xi_1 - \xi_2} P^{(w)} \\ &- \frac{\lambda(\lambda - 1) \xi_3^2}{\xi_1 - \xi_2} P^{\{sw\}} \end{aligned} \quad (26)$$

For further discussion, a most enlightening expression for (25) is obtained by separating the gauge-dependent parts

$$\begin{aligned} \mathcal{L}_{LD} &= \frac{1}{2} \tilde{h} a \left(\frac{1}{2} P^{(2)} - P^{(s)} \right) \square \tilde{h} + \frac{1}{2} \chi[\tilde{h}] \mathcal{G}^{(\tilde{h})} \chi[\tilde{h}] \\ &- \frac{1}{2} \tilde{\pi} \left[a \left(\frac{1}{2} P^{(2)} - P^{(s)} \right) \square + \frac{a^2}{2c} P^{(2)} + \frac{a^2}{2(3b + c)} P^{(s)} \right] \tilde{\pi} \\ &- \frac{1}{2} \chi[\tilde{\pi}] \mathcal{G}^{(\tilde{\pi})} \chi[\tilde{\pi}] + \mathcal{L}_s \end{aligned} \quad (27)$$

where

$$\mathcal{G}_{\alpha\beta}^{(\tilde{h})} = \xi_3 \theta_{\alpha\beta} + \xi_3 \omega_{\alpha\beta} = \xi_3 \eta_{\alpha\beta} \quad (28)$$

$$\mathcal{G}_{\alpha\beta}^{(\tilde{\pi})} = \xi_3 \frac{\xi_1 \square + \xi_3}{\xi_1 \square} \theta_{\alpha\beta} + \xi_3 \frac{(\xi_1 - \xi_2) \square + \xi_3}{(\xi_1 - \xi_2) \square} \omega_{\alpha\beta} \quad , \quad (29)$$

and the form of χ has been displayed in (8).

The physical meaning is now apparent: \tilde{h} and $\tilde{\pi}$ describe the massless and the massive DOF of the theory respectively. Notice that the gauge-invariant part of the kinetic term of $\tilde{\pi}$ reproduces that of the Fierz-Pauli theory [17].

The LD Lagrangian (27) thus obtained is non-local for arbitrary gauge parameters. However, we can have locality for a particular choice of parameters (see Appendix II). Even for this choice, an unpleasant feature of (27) is that the scalar subspaces S and W appear mixed as long as the transfer operator $P^{\{SW\}}$ occurs in N and $NM^{-1}N$.

3 Linearized theory and propagators

In order to avoid unessential complications due to the S - W mixing that obscures the identification of the propagating DOF, one may redefine the field $h_{\mu\nu}$ as

$$\hat{h}_{\mu\nu} = (Q^{-1})^{\rho\sigma} h_{\rho\sigma} \quad (30)$$

where

$$Q(\lambda) = P^{(2)} + P^{(1)} + \frac{2}{3}P^{(W)} - \frac{2(\lambda-1)}{9\lambda}P^{\{SW\}} \quad (31)$$

is invertible and becomes a numerical matrix for $\lambda = -2$, namely $Q(-2) = \bar{\eta} - \frac{1}{3}\bar{\bar{\eta}}$. This choice is compulsory if we wish to avoid polluting the source term with non-locality. The operator P transforms to

$$\hat{P} = QPQ = \hat{M}\square^2 + \hat{N}\square \quad , \quad (32)$$

where

$$\begin{aligned} \hat{M} &\equiv \frac{c}{2}P^{(2)} - \frac{1}{2}\xi_1 P^{(1)} \\ &\quad + \frac{4}{27}\frac{(\lambda-1)^2}{\lambda^2}2(3b+c)P^{(W)} - \frac{4}{27}\frac{(\lambda-1)^4}{\lambda^2}(\xi_1 - \xi_2)P^{(S)} \quad , \\ \hat{N} &\equiv \frac{1}{2}aP^{(2)} - \frac{4}{27}a\frac{(\lambda-1)^2}{\lambda^2}P^{(W)} - \frac{1}{2}\xi_3 P^{(1)} - \frac{4}{27}\frac{(\lambda-1)^4}{\lambda^2}\xi_3 P^{(S)} \end{aligned} \quad (33)$$

do not contain the operator $P^{\{SW\}}$ anylonger. Then equation (13) may be written as

$$\mathcal{L}_{HD} = \frac{1}{2}\hat{h}\square(\hat{M}\square + \hat{N})\hat{h} + \mathcal{L}_s \quad , \quad (34)$$

or, dropping total derivatives,

$$\mathcal{L}_{HD}^{(2)}[\hat{h}, \square \hat{h}] = \frac{1}{2}(\square \hat{h})\hat{M}(\square \hat{h}) + \frac{1}{2}\hat{h}\hat{N}(\square \hat{h}) + \hat{T}\hat{h} \quad . \quad (35)$$

The particle interpretation of (35) is now the central question. On one hand we can start from the HD theory (34) and, after inverting the projectors, obtain the quartic propagator

$$\begin{aligned} \Delta^{HD}[\hat{h}] = & \frac{2}{(c\square + a)\square}P^{(2)} + \frac{27}{4} \frac{\lambda^2}{(\lambda - 1)^2[2(3b + c)\square - a]\square}P^{(W)} \\ & - \frac{2}{(\xi_1\square + \xi_3)\square}P^{(1)} - \frac{27}{4} \frac{\lambda^2}{(\lambda - 1)^4[(\xi_1 - \xi_2)\square + \xi_3]\square}P^{(S)} \quad . \end{aligned} \quad (36)$$

On the other hand, the quadratic propagators arising from the LD theory (analogous of (25)) for the new *hat* fields are

$$\begin{aligned} \Delta^{LD}[\tilde{h}] = & \frac{2}{a\square}P^{(2)} - \frac{27}{4} \frac{\lambda^2}{(\lambda - 1)^2a\square}P^{(W)} \\ & - \frac{2}{\xi_3\square}P^{(1)} - \frac{27}{4} \frac{\lambda^2}{(\lambda - 1)^4\xi_3\square}P^{(S)} \quad , \\ \Delta^{LD}[\tilde{\pi}] = & -\frac{2c}{a(c\square + a)}P^{(2)} + \frac{27}{4} \frac{\lambda^2}{(\lambda - 1)^2a[2(3b + c)\square - a]}P^{(W)} \\ & + \frac{2}{\xi_3} \frac{\xi_1}{(\xi_1\square + \xi_3)}P^{(1)} + \frac{27}{4} \frac{\lambda^2}{(\lambda - 1)^4\xi_3} \frac{(\xi_1 - \xi_2)}{[(\xi_1 - \xi_2)\square + \xi_3]}P^{(S)} \quad . \end{aligned} \quad (37)$$

As expected, the LD quadratic propagators sum up to give the HD quartic one, namely

$$\Delta^{HD}[\hat{h}] = \Delta^{LD}[\tilde{h}] + \Delta^{LD}[\tilde{\pi}] \quad (38)$$

Notice that if we had not performed the Q -transformation, the propagators would have been

$$\begin{aligned} \Delta^{HD}[h] &= Q\Delta^{HD}[\hat{h}]Q \\ \Delta^{LD}[\tilde{h}] &= Q\Delta^{LD}[\tilde{h}]Q \\ \Delta^{LD}[\tilde{\pi}] &= Q\Delta^{LD}[\tilde{\pi}]Q \quad , \end{aligned} \quad (39)$$

with the mixing $P^{\{SW\}}$ occurring in them.

The DOF counting can be readily done on (37). Since we are dealing with a properly gauge fixed four-derivative theory, all the fields in $\hat{h}_{\mu\nu}$ do propagate and therefore we have a total of 20 DOF (10 massless and 10 massive). According to the dimensionality of the respective spin subspaces, they are distributed as 5, 3, 1 and 1 DOF for the spin-2, 1, 0_S and 0_W respectively, which sum up 10 DOF for the massless fields, and the same for the massive ones. In the (massless) \tilde{h} -sector, the spin-2 space contains the two DOF of the graviton plus three gauge DOF, and the remaining five

DOF are also gauge ones. The $\tilde{\pi}$ -sector describes the five DOF of a spin-2 poltergeist with (squared) mass a/c [13] [17], one physical scalar DOF with mass $-a/2(3b+c)$, three third ghosts with gauge-dependent masses ξ_3/ξ_1 and one third ghost with mass $\xi_3/(\xi_1 - \xi_2)$.

In the absence of gauge fixing only the projectors $P^{(2)}$ and $P^{(S)}$ are involved, and P^{inv} in (7) can be inverted only in the restricted spin subspace $2 \oplus S$. In principle, in that case one is left with eight DOF in the LD theory, namely, the massless graviton, the massive spin-2 poltergeist and the physical scalar. This is generally so as long as critical relationships between a , b and c are avoided [10] [13] that make the order-reducing procedure singular. In those cases some DOF may collapse and the theory may have fewer than eight DOF and/or larger symmetries. Fast DOF-counting recipes for gauge theories can be found in [18]. In ordinary two-derivative gravity, each of the four gauge-group local parameters of Diff-invariance accounts for the killing of *two* DOF, leaving the two DOF of the graviton out of the ten DOF of $h_{\mu\nu}$. In four-derivative gravity each gauge-group parameter instead kills *three* DOF so that the initial twenty DOF reduce to the eight DOF quoted above. The mechanism is well illustrated in four-derivative QED [15], where one initially has eight DOF and the gauge invariance suppress three of them, leaving one photon and one massive spin-1 poltergeist.

As detailed in Appendix II, the free part of the Q -transformed LD theory can be made local for a particular choice of gauge parameters. However, using $Q(\lambda)$, for $\lambda = b/(4b+c)$, moves the non-locality to the source term, namely to the interactions. This can also be avoided by requiring that $\lambda = -2$, in which case Q becomes a numerical matrix, but this gives rise to a condition on the parameters b and c of the starting Diff-invariant theory. Leaving aside the interpretation of such a restriction, this does not mean that we had anyway obtained a sum of independent Lagrangian theories for each (massless and massive, spin 2, 1, 0_S and 0_W) particle, notwithstanding the fact that the spin subspaces appear well separated. This is not possible, as illustrated for instance by the fact [19] that there is no second-order tensor local theory for spin-1 fields.

4 Faddeev-Popov compensating terms

As usual, the gauge fixing term (3) together with the compensating (HD) Faddeev-Popov Lagrangian can be expressed as a coboundary in the BRST cohomology, namely

$$\mathcal{L}_g + \mathcal{L}_{FP} = -s \left[\bar{C}^\alpha \mathcal{G}_{\alpha\beta} \chi^\beta[h] + \frac{1}{2} \bar{C}^\alpha \mathcal{G}_{\alpha\beta} \mathcal{B}^\beta \right] , \quad (40)$$

where \bar{C} are FP fermion ghosts and \mathcal{B} is an auxiliary commuting field.

In order to study the propagators of the new fields \bar{C} and \mathcal{B} , it suffices to consider the linearized objects

$$\begin{aligned}\chi^\beta[h] &= \chi^{\beta\mu\nu} h_{\mu\nu} \equiv (\eta^{\beta\mu} \partial^\nu - \lambda \eta^{\mu\nu} \partial^\beta) h_{\mu\nu} \\ \mathcal{G}_{\alpha\beta}^{(h)} &\equiv \xi_1 \eta_{\alpha\beta} \square - \xi_2 \partial_\alpha \partial_\beta + \xi_3 \eta_{\alpha\beta} \\ &= (\xi_1 \square + \xi_3) \theta_{\alpha\beta} + [(\xi_1 - \xi_2) \square + \xi_3] \omega_{\alpha\beta} \quad ,\end{aligned}\tag{41}$$

and the BRST symmetry given by the (linearized) Slavnov transformations

$$\begin{aligned}sh_{\mu\nu} &= \mathcal{D}_{\mu\nu,\alpha} C^\alpha \\ sC^\alpha &= 0 \\ s\bar{C}^\alpha &= \mathcal{B}^\alpha \\ s\mathcal{B}^\alpha &= 0 \quad ,\end{aligned}\tag{42}$$

where

$$\mathcal{D}_{\mu\nu,\beta} = \eta_{\mu\beta} \partial_\nu + \eta_{\nu\beta} \partial_\mu \tag{43}$$

is the gauge symmetry generator. With the diagonalization

$$\mathcal{B}^\alpha = B^\alpha - \chi^\alpha[h] \tag{44}$$

the linearized (40) becomes

$$\mathcal{L}_g^{(2)} + \mathcal{L}_{FP}^{HD} = \frac{1}{2} \chi^\alpha[h] \mathcal{G}_{\alpha\beta}^{(h)} \chi^\beta[h] + \bar{C}^\alpha \mathcal{G}_{\alpha\gamma} \chi^{\gamma\mu\nu} \mathcal{D}_{\mu\nu,\beta} C^\beta - \frac{1}{2} B^\alpha \mathcal{G}_{\alpha\beta}^{(h)} B^\beta \quad , \tag{45}$$

and one has

$$\begin{aligned}sh_{\mu\nu} &= \mathcal{D}_{\mu\nu,\alpha} C^\alpha \\ sC^\alpha &= 0 \\ s\bar{C}^\alpha &= B^\alpha - \chi^\alpha[h] \\ sB^\alpha &= \chi^{\alpha\mu\nu} \mathcal{D}_{\mu\nu,\beta} C^\beta \quad .\end{aligned}\tag{46}$$

Of course, they reflect the trivially Abelian gauge symmetry group G to which the infinitesimal diffeomorphisms reduce in the linearized theory. In the complete non-polynomial theory there are couplings between $h_{\mu\nu}$, C , \bar{C} and B , and the non-Abelian symmetry yields a more complicated set of S -transformations in which, for instance $sC^\alpha \neq 0$. One should also notice that the Q -transformation used in Section 3 is influent in the form of \mathcal{L}_{FP} , as can be checked by computing it with the corresponding operators $\hat{\mathcal{D}} = Q^{-1} \mathcal{D}$ and $\hat{\chi}^{\alpha\mu\nu} = (\chi Q)^{\alpha\mu\nu}$.

The fermionic sector of the FP Lagrangian above, namely

$$\mathcal{L}^{HD}[\bar{C}C] = \bar{C}^\alpha [(\xi_1 \square + \xi_3) \theta_{\alpha\beta} + 2(1 - \lambda)((\xi_1 - \xi_2) \square + \xi_3) \omega_{\alpha\beta}] C^\beta \tag{47}$$

is higher-derivative whereas, in contrast with ordinary second-order theories, now the auxiliary bosonic field B *does propagate* according to the Lagrangian

$$\mathcal{L}[B] = -\frac{1}{2} B^\alpha [(\xi_1 \square + \xi_3) \theta_{\alpha\beta} + [(\xi_1 - \xi_2) \square + \xi_3] \omega_{\alpha\beta}] B^\beta \tag{48}$$

which is already LD and always local. We can also perform an order-reduction of $\mathcal{L}^{HD}[\bar{C}C]$ (Appendix III), yielding

$$\begin{aligned} \mathcal{L}^{LD}[\bar{E}E\bar{F}F] &= \bar{E}^\alpha (\xi_3 \theta_{\alpha\beta} + 2(1-\lambda)\xi_3 \omega_{\alpha\beta}) \square E^\beta \\ &\quad - \bar{F}^\alpha \left(\frac{\xi_3}{\xi_1} (\xi_1 \square + \xi_3) \theta_{\alpha\beta} + \frac{2(1-\lambda)\xi_3}{\xi_1 - \xi_2} ((\xi_1 - \xi_2)\square + \xi_3) \omega_{\alpha\beta} \right) F^\beta, \end{aligned} \quad (49)$$

where $E^\alpha + F^\alpha = C^\alpha$ and $\bar{E}^\alpha + \bar{F}^\alpha = \bar{C}^\alpha$. The Lagrangian (49) is local for the same choice (see equations (102)-(104) in Appendix II) of gauge parameters that makes \mathcal{L}_{LD} in (25) local.

From (48) one directly reads

$$\Delta[B] = \frac{\theta}{\xi_1 \square + \xi_3} + \frac{\omega}{(\xi_1 - \xi_2)\square + \xi_3}, \quad (50)$$

whereas the HD (oriented) propagator

$$\begin{aligned} \Delta^{HD}[\bar{C}C] &= \frac{\theta}{(\xi_1 \square + \xi_3)\square} + \frac{\omega}{2(1-\lambda)((\xi_1 - \xi_2)\square + \xi_3)\square} \\ &= \frac{\theta}{\xi_3} \left(\frac{1}{\square} - \frac{\xi_1}{\xi_1 \square + \xi_3} \right) + \frac{\omega}{2(1-\lambda)\xi_3} \left(\frac{1}{\square} - \frac{\xi_1 - \xi_2}{(\xi_1 - \xi_2)\square + \xi_3} \right) \end{aligned} \quad (51)$$

obtained from (47) splits into the (also oriented) LD propagators

$$\Delta[\bar{E}E] = \frac{\theta}{\xi_3 \square} + \frac{\omega}{2(1-\lambda)\xi_3 \square} \quad (52)$$

$$\Delta[\bar{F}F] = -\frac{\xi_1}{\xi_3(\xi_1 \square + \xi_3)}\theta - \frac{(\xi_1 - \xi_2)}{2(1-\lambda)\xi_3((\xi_1 - \xi_2)\square + \xi_3)}\omega, \quad (53)$$

that can be derived from (49) as well.

In (52) one counts four FP *fermion* ghosts E and four \bar{E} with massless poles, giving eight *negative* loop contributions that compensate for the eight massless gauge ghosts quoted in Section 3. The compensation of the third ghosts contains non trivial features which are characteristic to the HD theories. From (53) one has that the FP *fermion* ghosts F and \bar{F} give six *negative* loop contributions with propagator poles at ξ_3/ξ_1 and two at $\xi_3/(\xi_1 - \xi_2)$. This over-compensates the (three plus one) third ghosts. Here is where the new *boson* FP ghosts B , propagating with (50), come to the rescue: they give three *positive* contributions with poles at ξ_3/ξ_1 and one at $\xi_3/(\xi_1 - \xi_2)$, thus providing the complete cancellation of ghost loop contributions.

This matching of the ghost masses is a consequence of the interplay of the order-reducing and BRST procedures. The master relationship is

$$\mathcal{G}^{(h)-1} = \mathcal{G}^{(\bar{h})-1} - \mathcal{G}^{(\bar{\pi})-1}, \quad (54)$$

where the massive poles of the third ghosts are displayed by $\mathcal{G}^{(h)-1}$ and $\mathcal{G}^{(\bar{\pi})-1}$, the latter also having massless zero-modes. To see it we find useful defining the differential operator

$$Z_\beta^\alpha \equiv \chi^{\alpha\mu\nu} \mathcal{D}_{\mu\nu,\beta} = \square \left[\theta_\beta^\alpha + 2(1-\lambda)\omega_\beta^\alpha \right], \quad (55)$$

and the differential kernels

$$K_{\alpha\beta}^{(i)} \equiv \mathcal{G}_{\alpha\gamma}^{(i)} Z_{\beta}^{\gamma} \quad (i = h, \tilde{h}, \tilde{\pi}) \quad (56)$$

occurring in the FP Lagrangians above and worked out in (47) and (49). The poles of the (massless) gauge ghosts lie in Z^{-1} whereas the operator $\mathcal{G}^{(\tilde{h})^{-1}}$ has no poles. From (54) and (56) it follows that

$$K^{(h)^{-1}} = K^{(\tilde{h})^{-1}} - K^{(\tilde{\pi})^{-1}} \quad (57)$$

which also reads $\Delta^{HD}[\bar{C}C] = \Delta[\bar{E}E] + \Delta[\bar{F}F]$. Thus the E -fields and the F -fields inherit the massless and the massive poles respectively. On the other hand $\Delta[B] = -\mathcal{G}^{(h)^{-1}}$, so that also the boson B -fields share the same massive poles.

The symmetries of $\mathcal{L}_g^{LD} + \mathcal{L}_{FP}^{LD}$ are not trivial either. The symmetry of the (invariant part of the) HD theory under the group G of the gauge variations

$$\delta h_{\mu\nu} = \mathcal{D}_{\mu\nu,\alpha} \varepsilon^{\alpha} \quad , \quad (58)$$

is inherited by the LD theory via (17) and (24) with the variations

$$\begin{aligned} \delta \tilde{h}_{\mu\nu} &= \left[\bar{\eta}_{\mu\nu}^{\rho\sigma} + \square(N^{-1}M)_{\mu\nu}^{\rho\sigma} \right] \mathcal{D}_{\rho\sigma,\alpha} \varepsilon^{\alpha} \\ &= \left[\frac{\xi_1 \square + \xi_3}{\xi_3} P_{\mu\nu}^{(1)\rho\sigma} + \frac{(\xi_1 - \xi_2) \square + \xi_3}{\xi_3} P_{\mu\nu}^{(W)\rho\sigma} \right] \mathcal{D}_{\rho\sigma,\alpha} \varepsilon^{\alpha} \quad , \\ \delta \tilde{\pi}_{\mu\nu} &= - \left[\square(N^{-1}M)_{\mu\nu}^{\rho\sigma} \right] \mathcal{D}_{\rho\sigma,\alpha} \varepsilon^{\alpha} \\ &= - \left[\frac{\xi_1 \square}{\xi_3} P_{\mu\nu}^{(1)\rho\sigma} + \frac{(\xi_1 - \xi_2) \square}{\xi_3} P_{\mu\nu}^{(W)\rho\sigma} \right] \mathcal{D}_{\rho\sigma,\alpha} \varepsilon^{\alpha} \quad , \end{aligned} \quad (59)$$

both depending on the same four gauge-group parameters $\varepsilon^{\alpha}(x)$. One may check that $\delta \tilde{h}_{\mu\nu} + \delta \tilde{\pi}_{\mu\nu} = \delta h_{\mu\nu}$. However, the *free* invariant part of the LD theory (25) exhibits a larger symmetry group, namely the fields \tilde{h} and $\tilde{\pi}$ may be given independent variations

$$\bar{\delta} \tilde{h}_{\mu\nu} = \mathcal{D}_{\mu\nu,\alpha} \varepsilon'^{\alpha} \quad (60)$$

$$\bar{\bar{\delta}} \tilde{\pi}_{\mu\nu} = \mathcal{D}_{\mu\nu,\alpha} \varepsilon''^{\alpha} \quad , \quad (61)$$

thus doubling the number of group parameters, with the original symmetry as a diagonal-like subgroup $G_1 \subset G \times G$, which is isomorphic to G [15]. One may then look at

$$\mathcal{L}_g^{LD}[\tilde{h}] \equiv \frac{1}{2} \chi[\tilde{h}] \mathcal{G}^{(\tilde{h})} \chi[\tilde{h}] \quad (62)$$

and

$$\mathcal{L}_g^{LD}[\tilde{\pi}] = -\frac{1}{2} \chi[\tilde{\pi}] \mathcal{G}^{(\tilde{\pi})} \chi[\tilde{\pi}] \quad , \quad (63)$$

occurring in (27), as separate gauge fixings for the symmetries (60) and (61) respectively, and wonder what happens with the whole BRST scheme.

The separate S -transformations would be

$$\begin{aligned}
s\tilde{h}_{\mu\nu} &= \mathcal{D}_{\mu\nu,\alpha}E^\alpha & s\tilde{\pi}_{\mu\nu} &= \mathcal{D}_{\mu\nu,\alpha}F^\alpha \\
sE^\alpha &= 0 & sF^\alpha &= 0 \\
s\bar{E}^\alpha &= B'^\alpha - \chi^\alpha[\tilde{h}] & s\bar{F}^\alpha &= B''^\alpha - \chi^\alpha[\tilde{\pi}] \\
sB'^\alpha &= \chi^{\alpha\mu\nu}\mathcal{D}_{\mu\nu,\beta}E^\beta & sB''^\alpha &= \chi^{\alpha\mu\nu}\mathcal{D}_{\mu\nu,\beta}F^\beta \quad ,
\end{aligned} \tag{64}$$

so we are led to write

$$\mathcal{L}_g + \mathcal{L}_{FP}^* = -s \left[\bar{E}^\alpha \mathcal{G}_{\alpha\beta}^{(\tilde{h})} \chi^\beta[\tilde{h}] + \frac{1}{2} \bar{E}^\alpha \mathcal{G}_{\alpha\beta}^{(\tilde{h})} \mathcal{B}'^\beta \right] \tag{65}$$

$$\begin{aligned}
& + s \left[\bar{F}^\alpha \mathcal{G}_{\alpha\beta}^{(\tilde{\pi})} \chi^\beta[\tilde{\pi}] + \frac{1}{2} \bar{F}^\alpha \mathcal{G}_{\alpha\beta}^{(\tilde{\pi})} \mathcal{B}''^\beta \right] \\
& = \frac{1}{2} \chi^\alpha[\tilde{h}] \mathcal{G}_{\alpha\beta}^{(\tilde{h})} \chi^\beta[\tilde{h}] - \frac{1}{2} \chi^\alpha[\tilde{\pi}] \mathcal{G}_{\alpha\beta}^{(\tilde{\pi})} \chi^\beta[\tilde{\pi}] \\
& + \bar{E}^\alpha \mathcal{G}_{\alpha\rho}^{(\tilde{h})} \chi^{\rho\mu\nu} \mathcal{D}_{\mu\nu,\beta} E^\beta - \bar{F}^\alpha \mathcal{G}_{\alpha\rho}^{(\tilde{\pi})} \chi^{\rho\mu\nu} \mathcal{D}_{\mu\nu,\beta} F^\beta \\
& - \frac{1}{2} B'^\alpha \mathcal{G}_{\alpha\beta}^{(\tilde{h})} B'^\beta + \frac{1}{2} B''^\alpha \mathcal{G}_{\alpha\beta}^{(\tilde{\pi})} B''^\beta
\end{aligned} \tag{66}$$

Thus (49) agrees with the fermionic sector of (66).

Equations (64) define two cohomologies $\{\bar{V}; s\}$ and $\{\bar{\bar{V}}; s\}$ with cohomological spaces $\bar{V} \equiv \{\tilde{h}, E, \bar{E}, B'\}$ and $\bar{\bar{V}} \equiv \{\tilde{\pi}, F, \bar{F}, B''\}$ respectively, both being copies of the original $V \equiv \{h, C, \bar{C}, B\}$ of (46) with boundary operator s . The polynomial (65) is then an exact cochain in the cohomology $\{\bar{V}; s\} \oplus \{\bar{\bar{V}}; s\} \equiv \{\bar{V} \oplus \bar{\bar{V}}; s \oplus s\}$.

The cohomology characterizing the HD theory appears as a subcohomology $\{V_1; s_1\}$ of the direct sum above. The subspace $V_1 \subset \bar{V} \oplus \bar{\bar{V}}$ is defined by

$$\begin{aligned}
\tilde{h}_{\mu\nu} &= \mathcal{O}'^{\rho\sigma}_{\mu\nu} h_{\rho\sigma} & \tilde{\pi}_{\mu\nu} &= \mathcal{O}''^{\rho\sigma}_{\mu\nu} h_{\rho\sigma} \\
E^\alpha &= \mathcal{O}'^\alpha_\beta C^\beta & F^\alpha &= \mathcal{O}''^\alpha_\beta C^\beta \\
\bar{E}^\alpha &= \mathcal{O}'^\alpha_\beta \bar{C}^\beta & \bar{F}^\alpha &= \mathcal{O}''^\alpha_\beta \bar{C}^\beta \\
B'^\alpha &= \mathcal{O}'^\alpha_\beta B^\beta & B''^\alpha &= \mathcal{O}''^\alpha_\beta B^\beta \quad ,
\end{aligned} \tag{67}$$

where

$$\mathcal{O}'^{\rho\sigma}_{\mu\nu} \equiv \bar{\eta}^{\rho\sigma}_{\mu\nu} + \square(N^{-1}M)^{\rho\sigma}_{\mu\nu} \tag{68}$$

$$\mathcal{O}''^{\rho\sigma}_{\mu\nu} \equiv -\square(N^{-1}M)^{\rho\sigma}_{\mu\nu} \tag{69}$$

$$\mathcal{O}'^\alpha_\beta \equiv \frac{\xi_1 \square + \xi_3}{\xi_3} \theta^\alpha_\beta + \frac{(\xi_1 - \xi_2) \square + \xi_3}{\xi_3} \omega^\alpha_\beta \tag{70}$$

$$\mathcal{O}''^\alpha_\beta \equiv -\left(\frac{\xi_1 \square}{\xi_3} \theta^\alpha_\beta + \frac{(\xi_1 - \xi_2) \square}{\xi_3} \omega^\alpha_\beta \right) \tag{71}$$

are invertible linear operators satisfying $\mathcal{O}' + \mathcal{O}'' = \delta$, and s_1 is the restriction to V_1 of $s \oplus s$. Then this subcohomology is nothing but the original one $\{V; s\}$ of the HD theory, since (67) defines an isomorphism $V \xrightarrow{u_1} V_1$ and s_1 becomes s , that is

$$\begin{aligned}
s\tilde{h}_{\mu\nu} + s\tilde{\pi}_{\mu\nu} &= sh_{\mu\nu} \\
sE^\alpha + sF^\alpha &= sC^\alpha \\
s\bar{E}^\alpha + s\bar{F}^\alpha &= s\bar{C}^\alpha \\
sB'^\alpha + sB''^\alpha &= sB^\alpha \quad ,
\end{aligned} \tag{72}$$

as a consequence of (67) and (68)-(71). In other words we have $\iota_1^{-1} \circ s \oplus s \circ \iota_1 = s$. Moreover, we recover the Lagrangian (48) for B , namely

$$\begin{aligned}\mathcal{L}^*[B'B''] &= -\frac{1}{2}B'^\alpha \mathcal{G}_{\alpha\beta}^{(\tilde{h})} B'^\beta + \frac{1}{2}B''^\alpha \mathcal{G}_{\alpha\beta}^{(\tilde{\pi})} B''^\beta \\ &= -\frac{1}{2}B^\alpha \mathcal{O}'_\alpha{}^\gamma \mathcal{G}_{\gamma\rho}^{(\tilde{h})} \mathcal{O}'_\beta{}^\rho B^\beta + \frac{1}{2}B^\alpha \mathcal{O}''_\alpha{}^\gamma \mathcal{G}_{\gamma\rho}^{(\tilde{\pi})} \mathcal{O}''_\beta{}^\rho B^\beta \\ &= -\frac{1}{2}B^\alpha \mathcal{G}_{\alpha\beta}^{(h)} B^\beta = \mathcal{L}[B]\end{aligned}\tag{73}$$

The subgroup $G_1 \subset G \times G$, associated to $\{V_1; s_1\}$ and isomorphic to G , is obtained by taking the group parameters ε' and ε'' as functions of four parameters ε by means of the equations

$$\begin{aligned}\mathcal{D}_{\mu\nu,\alpha}\varepsilon'^\alpha &= \mathcal{O}'_{\mu\nu}{}^{\rho\sigma} \mathcal{D}_{\rho\sigma,\alpha}\varepsilon^\alpha \\ \mathcal{D}_{\mu\nu,\alpha}\varepsilon''^\alpha &= \mathcal{O}''_{\mu\nu}{}^{\rho\sigma} \mathcal{D}_{\rho\sigma,\alpha}\varepsilon^\alpha\end{aligned}\tag{74}$$

These are derived by imposing the relations stemming from (59) on the otherwise independent variations (60) and (61), and yield

$$\begin{aligned}\varepsilon'^\alpha &= \mathcal{O}'_\beta{}^\alpha \varepsilon^\beta \\ \varepsilon''^\alpha &= \mathcal{O}''_\beta{}^\alpha \varepsilon^\beta\end{aligned}\tag{75}$$

so that $\varepsilon = \varepsilon' + \varepsilon''$. The subgroup G_1 is, by definition, a symmetry of the whole (non gauge-fixed) Lagrangian (and also separately of the interaction terms) since one has $\delta\tilde{h}_{\mu\nu} + \delta\tilde{\pi}_{\mu\nu} = \delta h_{\mu\nu}$, whereas $G \times G$ is broken by the interaction terms.

5 Conclusion

The interplay of gauge invariance and higher differential order in field theories gives rise to a remarkable diversity of particle-like states which are encrypted in the original field variables. In the four-derivative tensor theory of gravity here studied, the doubling of the initial conditions for the (fourth differential-order) equations of motion translates into a doubling of the effective number of particle-like DOF obeying second differential-order evolution equations. They describe physical (positive Fock-space norm) states together with an outburst of massless and massive ghostly states which are unphysical because of their negative norm and/or gauge dependence. Beyond the methodological interest, its analysis provides an enlarged context for the traditional gauge theories and BRST symmetries of physical relevance, which also enlightens the nature of some states already encountered in former classical works on higher-derivative gravity.

Four-derivative gravity is particularly interesting to study as long as the emphasis traditionally given to its applications has overlooked many details of its theoretical structure. Amongst the particle-like states of the gauge-fixed theory, there are physical ones (a massless graviton and one scalar, reminiscent of the Brans-Dicke field), a massive spin-2 gauge independent Weyl ghost (unphysical norm), and two families of

gauge-dependent fields: the usual massless gauge ghosts and the novel massive third ghosts. This elusive new breed of ghosts firstly arose in the exponentiation of the functional determinant of the differential operator $\mathcal{G}^{(h)}$.

In the presence of (generally HD) gauge fixing terms and of the corresponding compensating FP Lagrangian, the order-reducing procedure reveals remarkable features of the underlying BRST symmetry associated to the four-parameter gauge group G of infinitesimal diffeomorphisms. In parallel with the doubling of the fields, there is a doubling of the gauge symmetry of the *free* part of the (second-order) LD equivalent theory. Out of this $G \times G$ larger symmetry, both the interaction terms and the consistency of the BRST algebra select a diagonal-like subgroup G_1 , isomorphic to G , as the only symmetry of the complete LD theory, in agreement with the occurrence of Diff-invariance as the only symmetry of the starting HD theory. However, restricting ourselves to the free LD theory and considering its $G \times G$ symmetry, the LD gauge-dependent terms can be viewed as separate gauge fixings for both group factors. The (gauge-dependent) unphysical propagating DOF so introduced then appear as the respective gauge ghosts, which are massless for the first group factor and massive for the second one, thus giving further meaning to the famous third ghosts. Moreover, the separate symmetry of the gauge-independent part of the physical and poltergeist sectors of the LD theory illustrates also how their kinetic terms reproduce the structure of the Einstein's and Fierz-Pauli theory, thus describing (massless and massive) spin-2 fields respectively.

In correspondence with the appearance of a new class of massive gauge ghosts, the compensating FP Lagrangian also contains a greater number of propagating fields. These come from the HD doubling of the FP anticommuting fermion fields and from the boson fields, which are just auxiliary decoupled artifacts in ordinary two-derivative theories and now propagate and couple to the gauge-independent fields. The negative loop contributions of the massive fermion FP fields yield twice the amount needed to compensate for the third ghost loops, and it is just the positive contributions of the boson FP fields that provide the exact balance. This striking compensation mechanism, peculiar to HD gauge theories and easy to extrapolate to higher than four-derivative theories, illustrates well the power and richness of the BRST procedure. Of course, checking the exact cancellation of ghost loop contributions would require considering the actual residues of the propagators and vertex couplings arising in the complete non-polynomial theory, a task which is beyond the purposes of this work.

A final comment on locality is in order. From a HD local theory, the order-reducing procedure leads to an equivalent two-derivative theory. For scalar theories, the LD counterpart is directly local [16]. In gauge theories of vector fields there is always a choice of the gauge fixing parameters for which it is also local [15]. For tensor fields, the example studied in this paper tells us that obtaining an equivalent LD theory with independent free Lagrangians for the different spin states is not compatible with locality, although one comes close to this goal by suitably picking the gauge parameters. This obstruction is related to the more complex structure of the constraints of the tensor field theories, like the one that prevents having a tensor local theory of second differential-order for spin-1 fields.

Appendix I

We use the notations

$$\begin{aligned} g_{\mu\nu} &= \eta_{\mu\nu} + h_{\mu\nu} \\ A^\mu &\equiv \partial_\nu h^{\mu\nu} \\ h &\equiv h_\mu^\mu \\ X_{(\mu\nu)} &\equiv X_{\{\mu\nu\}} \equiv X_{\mu\nu} + X_{\nu\mu} \quad , \end{aligned}$$

and the Minkowsky metric is $\eta_{\mu\nu} = \text{diag}(+ - - -)$.

The spin projectors are

$$P_{\mu\nu,\rho\sigma}^{(2)} = \frac{1}{2}\theta_{\mu(\rho}\theta_{\nu\sigma)} - \frac{1}{3}\theta_{\mu\nu}\theta_{\rho\sigma} \quad (77)$$

$$P_{\mu\nu,\rho\sigma}^{(1)} = \frac{1}{2}\theta_{\{\mu(\rho}\omega_{\nu\}\sigma)} \quad (78)$$

$$P_{\mu\nu,\rho\sigma}^{(S)} = \frac{1}{3}\theta_{\mu\nu}\theta_{\rho\sigma} \quad (79)$$

$$P_{\mu\nu,\rho\sigma}^{(W)} = \omega_{\mu\nu}\omega_{\rho\sigma} \quad (80)$$

They are symmetric under the interchanges

$$\mu \leftrightarrow \nu \quad , \quad \rho \leftrightarrow \sigma \quad , \quad \mu\nu \leftrightarrow \rho\sigma \quad , \quad (81)$$

idempotent, orthogonal to each other, and sum up to the identity operator in the space of symmetric two-tensors, namely

$$\bar{\eta}_{\mu\nu,\rho\sigma} \equiv \frac{1}{2}\eta_{\mu(\rho}\eta_{\nu\sigma)} = (P^{(2)} + P^{(1)} + P^{(S)} + P^{(W)})_{\mu\nu,\rho\sigma} \quad . \quad (82)$$

These projectors are constructed using the transverse and longitudinal projectors for vectors fields

$$\theta_{\mu\nu} = \eta_{\mu\nu} - \frac{\partial_\mu \partial_\nu}{\square} \quad (83)$$

$$\omega_{\mu\nu} = \frac{\partial_\mu \partial_\nu}{\square} \quad . \quad (84)$$

We also use the transfer operators

$$P_{\mu\nu,\rho\sigma}^{(SW)} = \theta_{\mu\nu}\omega_{\rho\sigma} \quad (85)$$

$$P_{\mu\nu,\rho\sigma}^{(WS)} = \omega_{\mu\nu}\theta_{\rho\sigma} \quad (86)$$

from which we define

$$P^{\{SW\}} = P^{\{WS\}} \equiv P^{(SW)} + P^{(WS)} \quad . \quad (87)$$

They have non-zero products

$$P^{(SW)}P^{(WS)} = 3P^{(S)} \quad (88)$$

$$P^{(WS)}P^{(SW)} = 3P^{(W)} \quad (89)$$

$$P^{(S)}P^{(SW)} = P^{(SW)}P^{(W)} = P^{(SW)} \quad (90)$$

$$P^{(W)}P^{(WS)} = P^{(WS)}P^{(S)} = P^{(WS)} \quad (91)$$

$$P^{\{SW\}}P^{\{SW\}} = 3(P^{(S)} + P^{(W)}) \quad (92)$$

$$P^{(S)}P^{\{SW\}} = P^{\{SW\}}P^{(W)} = P^{(SW)} \quad (93)$$

$$P^{(W)}P^{\{SW\}} = P^{\{SW\}}P^{(S)} = P^{(WS)} \quad (94)$$

We define also

$$\bar{\bar{\eta}}_{\mu\nu,\rho\sigma} \equiv \eta_{\mu\nu}\eta_{\rho\sigma} = 3P^{(S)} + P^{(W)} + P^{\{SW\}} \quad (95)$$

Here we collect some formulae which are useful for dealing with combinations of the operators above. Inverse:

$$\begin{aligned} \mathcal{M} &= \lambda_2 P^{(2)} + \lambda_1 P^{(1)} + \lambda_S P^{(S)} + \lambda_W P^{(W)} + \lambda_{SW} P^{\{SW\}} \\ \mathcal{M}^{-1} &= \frac{1}{\lambda_2} P^{(2)} + \frac{1}{\lambda_1} P^{(1)} + \frac{\lambda_W}{\lambda_S \lambda_W - 3\lambda_{SW}^2} P^{(S)} + \frac{\lambda_S}{\lambda_S \lambda_W - 3\lambda_{SW}^2} P^{(W)} \\ &\quad - \frac{\lambda_{SW}}{\lambda_S \lambda_W - 3\lambda_{SW}^2} P^{\{SW\}} \quad . \end{aligned}$$

When computing symmetric products like (26), operators in the subspace $S \oplus W$ of the form

$$\Omega(\tau_S, \tau_W, \tau_{SW}) = \tau_S P^{(S)} + \tau_W P^{(W)} + \tau_{SW} P^{\{SW\}} \quad (96)$$

occur, for which one has the product law

$$\begin{aligned} \Omega(\tau_S, \tau_W, \tau_{SW})\Omega(\lambda_S, \lambda_W, \lambda_{SW})\Omega(\tau_S, \tau_W, \tau_{SW}) &= \\ \Omega(\tau_S^2 \lambda_S + 3\tau_{SW}^2 \lambda_W + 6\tau_S \tau_{SW} \lambda_{SW} \quad , & \\ \tau_W^2 \lambda_W + 3\tau_{SW}^2 \lambda_S + 6\tau_W \tau_{SW} \lambda_{SW} \quad , & \\ \tau_S \tau_W \lambda_{SW} + \tau_S \tau_{SW} \lambda_S + \tau_{SW} \tau_W \lambda_W + 3\tau_{SW}^2 \lambda_{SW}) & \end{aligned} \quad (97)$$

A basis of zeroth differential-order local operators with the symmetries (81) is provided by $\bar{\eta}$ and $\bar{\bar{\eta}}$. Local second-order operators can be expanded in the basis

$$\begin{aligned} \mathcal{C}_1 \square &:= \left(\frac{1}{2} P^{(2)} - P^{(S)} \right) \square \\ \mathcal{C}_2 \square &:= \left(\frac{1}{2} P^{(1)} + 3P^{(S)} \right) \square \\ \mathcal{C}_3 \square &:= \left(P^{\{SW\}} + 6P^{(S)} \right) \square \\ \mathcal{C}_4 \square &:= \left(P^{(W)} - 3P^{(S)} \right) \square \end{aligned} \quad (98)$$

Thus, a general local LD operator has the form

$$\Omega^{LD} = \sum_{i=1}^4 \alpha_i \mathcal{C}_i \square + a_1 \bar{\eta} + a_2 \bar{\bar{\eta}} \quad . \quad (99)$$

Einstein's (linearized) theory displays the operator \mathcal{C}_1 . Fierz-Pauli's has the same kinetic term and a mass term built with $\bar{\eta} - \bar{\bar{\eta}}$. When using the field basis obtained by $Q(\lambda)$ -transforming the theory, this kinetic term displays the operator

$$Q(\lambda) \mathcal{C}_1 \square Q(\lambda) = \left(\frac{1}{2} P^{(2)} - \frac{4}{27} \frac{(\lambda - 1)^2}{\lambda^2} P^{(W)} \right) \square \quad (100)$$

For $\lambda = -2$ it becomes local again, namely $\left(\frac{1}{2} P^{(2)} - \frac{1}{3} P^{(W)} \right) \square = \mathcal{C}_1 \square - \frac{1}{3} \mathcal{C}_3 \square$ which describes (linearized) gravity as properly as the original operator $\mathcal{C}_1 \square$ did.

Appendix II

The kinetic terms for \tilde{h} and $\tilde{\pi}$ in (25) contain the operator $N \square$ which is local for arbitrary gauge parameters. In fact one has that

$$N = a \mathcal{C}_1 - \xi_3 \mathcal{C}_2 - \lambda(\lambda - 1) \xi_3 \mathcal{C}_3 - \xi_3(\lambda - 1)^2 \mathcal{C}_4 \quad (101)$$

However, the “mass term” $-\frac{1}{2} \tilde{\pi} N M^{-1} N \tilde{\pi}$ is local only for a choice of gauge parameters obeying the conditions

$$\xi_1 = -c \frac{\xi_3^2}{a^2} \quad , \quad (102)$$

$$\frac{a^2}{2c} \frac{\xi_1 - \xi_2}{\xi_3^2} = -\frac{3b + c}{4b + c} \quad , \quad (103)$$

$$\lambda = \frac{b}{4b + c} \quad , \quad (104)$$

that are obtained by requiring $N M^{-1} N$ to be a linear combination of $\bar{\eta}$ and $\bar{\bar{\eta}}$. This leaves one of the parameters ξ still arbitrary. This same conditions make the fermion Lagrangian (48) local.

In view of the conditions above, a theory in which $4b + c = 0$ does not have a local LD equivalent. But this case was critical already for the complete Diff-invariant theory since it is not regular in R and a general-covariant Legendre transform (see equation (8) of [13]) cannot be performed. With gauge fixing terms and for the linearized field, the (just Lorentz-covariant) Legendre transform can always be carried out (equation (17) is not singular for $4b + c = 0$) and we instead have non-locality of the LD theory.

When we consider the Q -transformed theory, the potentially non-local operator is

$$\begin{aligned}\hat{N}\hat{M}^{-1}\hat{N} &= \frac{a^2}{2c}P^{(2)} + \frac{4}{27}\frac{(\lambda-1)^2}{\lambda^2}\frac{a^2}{2(3b+c)}P^{(W)} \\ &- \frac{\xi_3^2}{2\xi_1}P^{(1)} - \frac{4}{27}\frac{(\lambda-1)^4}{\lambda^2}\frac{\xi_3^2}{\xi_1-\xi_2}P^{(S)} \quad .\end{aligned}\quad (105)$$

As explained before, we must take $\lambda = -2$ in order to keep the locality of the source term. In that case $\hat{N}\square$ remains local. Requiring locality for $\hat{N}\hat{M}^{-1}\hat{N}$ leads to

$$\xi_1 = -\frac{1}{5}\xi_2 \quad (106)$$

$$\xi_3^2 = \frac{a^2}{5c}\xi_2 \quad (107)$$

$$c = -\frac{9}{2}b \quad (108)$$

These conditions are the same found before, but now (104) yields a condition, namely (108), on the parameters of the original gauge-invariant theory.

Appendix III

We briefly outline the order-reduction of the higher-derivative FP Lagrangian for anticommuting fields

$$\mathcal{L}_{HD} = \bar{C}_\mu (\square(a_1\square + b_1)\theta^{\mu\nu} + \square(a_2\square + b_2)\omega^{\mu\nu}) C_\nu + \bar{\zeta}^\mu C_\mu + \bar{C}_\mu \zeta^\mu \quad (109)$$

where ζ and $\bar{\zeta}$ are external sources which are also anticommuting. Dropping total spacetime derivatives, conjugate momenta may be defined as the left derivatives

$$\mathcal{P}^\mu = \frac{\partial^L \mathcal{L}_{HD}}{\partial \square \bar{C}_\mu} = \mathcal{M}^{\mu\nu} \square C_\nu + \frac{1}{2} \mathcal{N}^{\mu\nu} C_\nu \quad (110)$$

$$\bar{\mathcal{P}}^\mu = \frac{\partial^L \mathcal{L}_{HD}}{\partial \square C_\mu} = -\mathcal{M}^{\mu\nu} \square \bar{C}_\nu - \frac{1}{2} \mathcal{N}^{\mu\nu} \bar{C}_\nu \quad (111)$$

where

$$\mathcal{M} \equiv a_1\theta + a_2\omega \quad , \quad \mathcal{N} \equiv b_1\theta + b_2\omega \quad , \quad (112)$$

from which

$$\square C_\mu = \mathcal{M}_{\mu\nu}^{-1} \left(\mathcal{P}^\nu - \frac{1}{2} \mathcal{N}^{\nu\rho} C_\rho \right) \quad (113)$$

$$\square \bar{C}_\mu = -\mathcal{M}_{\mu\nu}^{-1} \left(\bar{\mathcal{P}}^\nu + \frac{1}{2} \mathcal{N}^{\nu\rho} \bar{C}_\rho \right) \quad (114)$$

Then the “Hamiltonian” is

$$\begin{aligned}\mathcal{H} &\equiv (\Box \bar{C})\mathcal{P} + (\Box C)\mathcal{P} - \mathcal{L} \\ &= -\left(\bar{\mathcal{P}} + \frac{1}{2}\mathcal{N}\bar{C}\right)\mathcal{M}^{-1}\left(\mathcal{P} - \frac{1}{2}\mathcal{N}C\right) - \bar{\zeta}^\mu C_\mu - \bar{C}_\mu \zeta^\mu\end{aligned}\quad (115)$$

With the field redefinition

$$\begin{aligned}C &= E + F & \bar{C} &= \bar{E} + \bar{F} \\ \mathcal{P} &= \frac{1}{2}\mathcal{N}(E - F) & \bar{\mathcal{P}} &= \frac{1}{2}\mathcal{N}(\bar{F} - \bar{E})\end{aligned}\quad (116)$$

the Helmholtz Lagrangian

$$\mathcal{L}_H \equiv (\Box C)\bar{\mathcal{P}} + (\Box \bar{C})\mathcal{P} - \mathcal{H} \quad (117)$$

becomes

$$\begin{aligned}\mathcal{L}_{LD} &= \bar{E}\mathcal{N}\Box E - \bar{F}\left(\mathcal{N}\Box + \mathcal{N}\mathcal{M}^{-1}\mathcal{N}\right)F \\ &\quad + \bar{\zeta}(E + F) + (\bar{E} + \bar{F})\zeta \quad .\end{aligned}\quad (118)$$

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